Linear Algebra I

20/03/2014, Thursday, 14:00-16:00

You are **NOT** allowed to use any type of calculators.

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1 (8+7+7=22 pts) **Inner product spaces**

Consider the vector space $\mathbb{R}^{2\times 2}$. Let

$$\langle A, B\rangle = \mathrm{tr}(A^T B)$$

where tr denotes the sum of the diagonal elements.

a. Show that $\langle A, B\rangle$ is an inner product.

b. Find the distance between the matrices $\begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix}$ and $\begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix}$.

c. Find the angle between the matrices $\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$.

Required Knowledge: **definition of inner product spaces.**

Solution:

1a: We need to show that

i. $\langle A, A\rangle \geqslant 0$ for all $A \in \mathbb{R}^{2\times 2}$, and $\langle A, A\rangle = 0$ if and only if $A = 0$,

ii. $\langle A, B\rangle = \langle B, A\rangle$ for all $A, B \in \mathbb{R}^{2\times 2}$,

iii. $\langle \alpha A + \beta B, C\rangle = \alpha\langle A, C\rangle + \beta\langle B, C\rangle$ for all $A, B, C \in \mathbb{R}^{2\times 2}$ and $\alpha, \beta \in \mathbb{R}$.

To show (i), let $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Then, we have

$$A^T A = \begin{bmatrix} a & c \\ b & d \end{bmatrix} \begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} a^2 + c^2 & ab + cd \\ ab + cd & b^2 + d^2 \end{bmatrix}.$$

This means that

$$\langle A, A\rangle = a^2 + c^2 + b^2 + d^2 \geqslant 0$$

for all $A \in \mathbb{R}^{2\times 2}$. Moreover, we have

$$a^2 + c^2 + b^2 + d^2 = 0 \quad \text{if and only if} \quad a = b = c = d = 0.$$

In other words,

$$\langle A, A\rangle = 0 \quad \text{if and only if} \quad A = 0.$$

To show (ii), let

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \qquad B = \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix}.$$

Note that

$$A^T B = \begin{bmatrix} a_{11} & a_{21} \\ a_{12} & a_{22} \end{bmatrix} \begin{bmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{bmatrix} = \begin{bmatrix} a_{11}b_{11} + a_{21}b_{21} & a_{11}b_{12} + a_{21}b_{22} \\ a_{12}b_{11} + a_{22}b_{21} & a_{12}b_{12} + a_{22}b_{22} \end{bmatrix}$$

and

$$B^TA = \begin{bmatrix} b_{11} & b_{21} \\ b_{12} & b_{22} \end{bmatrix} \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} = \begin{bmatrix} b_{11}a_{11} + b_{21}a_{21} & b_{11}a_{12} + b_{21}a_{22} \\ b_{12}a_{11} + b_{22}a_{21} & b_{12}a_{12} + b_{22}a_{22} \end{bmatrix}.$$

Then, we have

$$\mathrm{tr}(A^TB) = \mathrm{tr}(B^TA) = a_{11}b_{11} + a_{21}b_{21} + a_{12}b_{12} + a_{22}b_{22}.$$

This means that

$$\langle A, B\rangle = \langle B, A\rangle$$

for all A, $B \in \mathbb{R}^{2\times 2}$.

To show (iii), note that

$$\langle \alpha A + \beta B, C\rangle = \mathrm{tr}\big((\alpha A + \beta B)^T C\big) = \mathrm{tr}(\alpha A^T C) + \mathrm{tr}(\beta B^T C)$$

since $\mathrm{tr}(M + N) = \mathrm{tr}(M) + \mathrm{tr}(N)$. Now, it follows from the fact that $\mathrm{tr}(\mu M) = \mu\mathrm{tr}(M)$ that we have

$$\langle \alpha A + \beta B, C\rangle = \mathrm{tr}(\alpha A^T C) + \mathrm{tr}(\beta B^T C) = \alpha\mathrm{tr}(A^T C) + \beta\mathrm{tr}(B^T C) = \alpha\langle A, C\rangle + \beta\langle B, C\rangle.$$

for all A, B, $C \in \mathbb{R}^{2\times 2}$ and α, $\beta \in \mathbb{R}$.

1b: The distance of two matrices A and B is defined by

$$\|A - B\| = \langle A - B, A - B\rangle^{\frac{1}{2}}.$$

Hence, we have

$$\begin{aligned} \| \begin{bmatrix} 1 & 2 \\ 1 & 0 \end{bmatrix} - \begin{bmatrix} 3 & 3 \\ 1 & 2 \end{bmatrix} \|^2 = \| \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} \|^2 &= \mathrm{tr}\left(\begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix}^T \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} \right) \\ &= \mathrm{tr}\left(\begin{bmatrix} -2 & 0 \\ -1 & -2 \end{bmatrix} \begin{bmatrix} -2 & -1 \\ 0 & -2 \end{bmatrix} \right) = \mathrm{tr}\left(\begin{bmatrix} 4 & 2 \\ 2 & 5 \end{bmatrix} \right) = 9. \end{aligned}$$

Therefore, the distance between these matrices is 3.

1c: The angle two matrices A and B is defined by

$$\cos\theta = \frac{\langle A, B\rangle}{\|A\|\|B\|}.$$

Note that

$$\langle \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \rangle = \mathrm{tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) = \mathrm{tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right) = \mathrm{tr}\left(\begin{bmatrix} 2 & 0 \\ 2 & 0 \end{bmatrix} \right) = 2,$$

$$\| \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \| = \mathrm{tr}\left(\begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \right)^{\frac{1}{2}} = \mathrm{tr}\left(\begin{bmatrix} 2 & 2 \\ 2 & 2 \end{bmatrix} \right)^{\frac{1}{2}} = 2,$$

and

$$\| \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \| = \mathrm{tr}\left(\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}^T \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right)^{\frac{1}{2}} = \mathrm{tr}\left(\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \right)^{\frac{1}{2}} = \mathrm{tr}\left(\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix} \right)^{\frac{1}{2}} = 2.$$

Therefore, we get

$$\cos\theta = \frac{2}{2\cdot 2} = \frac{1}{2}$$

which means that $\theta = \frac{\pi}{3}$.

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a. Find an orthogonal matrix that diagonalizes the matrix $\begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix}$.

b. Without finding its eigenvalues, determine whether or not the matrix $\begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix}$ is unitarily diagonalizable.

REQUIRED KNOWLEDGE: **diagonalization, normal matrices.**

SOLUTION:

2a: We begin with finding the eigenvalues. Note that

$$\det\left(\begin{bmatrix} 3-\lambda & 0 & -2 \\ 0 & 3-\lambda & 0 \\ -2 & 0 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)\det\left(\begin{bmatrix} 3-\lambda & -2 \\ -2 & 3-\lambda \end{bmatrix}\right) = (3-\lambda)\big((3-\lambda)^2-4\big)$$
$$= (3-\lambda)(3-\lambda-2)(3-\lambda+2).$$

Therefore, the eigenvalues are $\lambda_1 = 1$, $\lambda_2 = 3$, and $\lambda_3 = 5$. Now, we proceed with finding eigenvectors.

For $\lambda_1 = 1$, we have

$$\begin{bmatrix} 2 & 0 & -2 \\ 0 & 2 & 0 \\ -2 & 0 & 2 \end{bmatrix} x_1 = 0 \quad \Longrightarrow \quad x_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}.$$

For $\lambda_2 = 3$, we have

$$\begin{bmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ -2 & 0 & 0 \end{bmatrix} x_2 = 0 \quad \Longrightarrow \quad x_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

For $\lambda_3 = 5$, we have

$$\begin{bmatrix} -2 & 0 & -2 \\ 0 & -2 & 0 \\ -2 & 0 & -2 \end{bmatrix} x_3 = 0 \quad \Longrightarrow \quad x_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}.$$

Since the matrix we deal with is symmetric and eigenvectors corresponding to distinct eigenvalues of a symmetric matrix are orthogonal each other, the vectors x_1, x_2, and x_3 are mutually orthogonal. Then, we only need to normalise those vectors in order to obtain an orthogonal diagonalizer. Thus, we get

$$\begin{bmatrix} 3 & 0 & -2 \\ 0 & 3 & 0 \\ -2 & 0 & 3 \end{bmatrix} \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} = \begin{bmatrix} \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \\ 0 & 1 & 0 \\ \frac{1}{\sqrt{2}} & 0 & -\frac{1}{\sqrt{2}} \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 5 \end{bmatrix}.$$

2b: A complex matrix is unitarily diagonalizable if and only if it is normal. Note that

$$\begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix}^H \begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix} = \begin{bmatrix} -i & 1 & -1 \\ -1 & i & 1 \\ 1 & -1 & -i \end{bmatrix} \begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1-2i \\ -1 & 3 & -1 \\ -1+2i & -1 & 3 \end{bmatrix}$$

and that

$$\begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix} \begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix}^H = \begin{bmatrix} i & -1 & 1 \\ 1 & -i & -1 \\ -1 & 1 & i \end{bmatrix} \begin{bmatrix} -i & 1 & -1 \\ -1 & i & 1 \\ 1 & -1 & -i \end{bmatrix} = \begin{bmatrix} 3 & -1 & -1-2i \\ -1 & 3 & -1 \\ -1+2i & -1 & 3 \end{bmatrix}.$$

Therefore, the matrix we deal with is normal and thus unitarily diagonalizable.

3 (15+7=22 pts) **Singular value decomposition**

a. Compute the singular value decomposition of the matrix $M = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}$.

b. Find the closest (with respect to Frobenius norm) matrix of rank 2 to M.

Required Knowledge:**singular value decomposition, lower rank approximation.**

Solution:

3a: Note that

$$M^TM = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix}^T \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 16 \end{bmatrix}.$$

Then, we can conclude that the eigenvalues are given by $\lambda_1 = 16$, $\lambda_2 = 9$, $\lambda_3 = 5$, and $\lambda_4 = 0$. Hence, the singular values are $\sigma_1 = 4$, $\sigma_2 = 3$, $\sigma_3 = \sqrt{5}$, and $\sigma_4 = 0$. Since M^TM is already diagonal, we only need to change the oder of diagonal elements in order to diagonalize it with respect to the order of the eigenvalues we have. Note that

$$M^TM = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 16 & 0 & 0 & 0 \\ 0 & 9 & 0 & 0 \\ 0 & 0 & 5 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$

is diagonalization of M^TM by an orthogonal matrix. As such, we can take

$$V = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Now, we have

$$u_1 = \frac{1}{\sigma_1} M v_1 = \frac{1}{4} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix},$$

$$u_2 = \frac{1}{\sigma_2} M v_2 = \frac{1}{3} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 0 \end{bmatrix},$$

$$u_3 = \frac{1}{\sigma_3} M v_3 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix} = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 0 \\ 0 \\ 2 \end{bmatrix}.$$

Finally, we need to determine an orthonormal basis for the null space of M^T. Note that

$$0 = M^T x = \begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 4 & 0 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix} = \begin{bmatrix} x_1 + 2x_4 \\ 3x_3 \\ 0 \\ 4x_2 \end{bmatrix}.$$

This leads to

$$u_4 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Consequently, we have the following singular value decomposition:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 2 & 0 & 0 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & \sqrt{5} & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}.$$

3b: The closest rank 2 approximation can be found as:

$$\tilde{M} = \begin{bmatrix} 0 & 0 & \frac{1}{\sqrt{5}} & -\frac{2}{\sqrt{5}} \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \frac{2}{\sqrt{5}} & \frac{1}{\sqrt{5}} \end{bmatrix} \begin{bmatrix} 4 & 0 & 0 & 0 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix} = \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 4 \\ 0 & 3 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

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a. Let A be a nonsingular matrix. Show that if λ is an eigenvalue of A then $\frac{1}{\lambda}$ is an eigenvalue of A^{-1}.

b. Show that the determinant of an orthogonal matrix is either -1 or 1.

c. Show that eigenvalue of an orthogonal matrix must have modulus 1. [**Hint:** Modulus of a complex number z is defined by $\|z\| = (\bar{z}z)^{1/2}$.]

d. Let M be a normal matrix. Show that if all eigenvalues are equal to 1 then $M = I$.

Required Knowledge: **eigenvalues, orthogonal matrices, normal matrices.**

Solution:

4a: Let (λ, x) be an eigenpair of A, that is

$$Ax = \lambda x.$$

Since A is nonsingular, we get

$$\begin{aligned} A^{-1}Ax &= \lambda A^{-1}x \\ x &= \lambda A^{-1}x \\ \frac{1}{\lambda}x &= A^{-1}x \end{aligned}$$

which proves the claim.

4b: If U is orthogonal, then $U^TU = I$. Then, we have

$$1 = \det(U^TU) = \det(U^T)\det(U) = \det(U)^2$$

as determinant is invariant under transposition. Hence, we get $\det(U) = \mp 1$.

4c: Let U be an orthogonal matrix and (λ, x) be an eigenpair of U. Then, we have

$$Ux = \lambda x.$$

This means that

$$x^HU^H = \bar{\lambda}x^H$$

and hence

$$x^HU^HUx = \bar{\lambda}\lambda x^Hx.$$

Since U is orthogonal, we have $U^H = U^T$ and hence $U^HU = I$. Therefore, we get

$$x^Hx = \bar{\lambda}\lambda x^Hx$$

which proves the claim.

4d: Since M is normal, it can be unitarily diagonalizable, that is

$$M = U^HDU$$

where U is a unitary matrix and D is diagonal carrying the eigenvalues of M on the diagonal. If all eigenvalues of M are equal to 1, then $D = I$ and hence $M = U^HU = I$.